A note on the Painlevé property of coupled KdV equations

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Abstract

We prove that one system of coupled KdV equations, recently claimed by Hirota, Hu and Tang [J. Math. Anal. Appl. 288:326–348 (2003)] to pass the Painlevé test for integrability, actually fails the test at the highest resonance of the generic branch.

Introduction

In Section 6 of their recent work [1], Hirota, Hu and Tang reported that the system of coupled KdV equations

$$\frac{\partial u_i}{\partial t} + 6a\left(\sum_{k=1}^{N} u_k\right)\frac{\partial u_i}{\partial x} + 6(1-a)\left(\sum_{k=1}^{N}\frac{\partial u_k}{\partial x}\right)u_i + \frac{\partial^3 u_i}{\partial x^3} = 0, \qquad i = 1, 2, \ldots, N \qquad (N \geq 2) \tag{1}$$

passes the Painlevé test for integrability iff the parameter a is equal to 1 or 1/2 or 3/2. The authors of [1] pointed out that the cases $a = 1$ and $a = 1/2$ of (1) correspond to integrable systems of coupled KdV equations, whereas the problem of integrability of (1) with $a = 3/2$ remains open. In the present short note, we show that the system (1) with $a = 3/2$ actually does not pass the Painlevé test, and its integrability should not be expected therefore.

Singularity analysis

First of all, let us notice that the N-component system (1) can be studied in the following form of two coupled KdV equations:

$$v_t + 6vv_x + v_{xxx} = 0, \qquad w_t + 6avw_x + 6(1-a)wv_x + w_{xxx} = 0, \tag{2}$$

where $v = \sum_{k=1}^{N} u_k$, w is any one of the N components $u_1, \dots, u_N$, and the subscripts x and t denote partial derivatives. This equivalence between (1) and (2) means that any solution v of the first equation of (2) and any $N-1$ solutions w (not necessarily different, but corresponding to this v) of the second equation of (2) determine a solution $u_1, \dots, u_N$ of the system (1), and vice versa.

Now, setting $a = 3/2$ in (2) and starting the Weiss–Kruskal algorithm of singularity analysis [2, 3], we use the expansions $v = v_0(t)\phi^\alpha + \cdots + v_r(t)\phi^{r+\alpha} + \cdots$ and $w = w_0(t)\phi^\beta + \cdots + w_r(t)\phi^{r+\beta} + \cdots$ with $\phi_x(x,t) = 1$, and determine branches (i.e. admissible choices of α and β) together with corresponding positions r of resonances (where arbitrary functions of t can enter the expansions). The exponents α and β and positions of resonances turn out to be integer in all branches. In what follows, we only consider the generic singular branch, where $\alpha = \beta = -2$, $v_0 = -2$, $w_0(t)$ is arbitrary, and $r = -1, 0, 1, 4, 6, 8$; this branch describes the singular behavior of the general solution.

Substituting the expansions

$$v = \sum_{n=0}^{\infty} v_n(t)\phi^{n-2}, \qquad w = \sum_{n=0}^{\infty} w_n(t)\phi^{n-2} \tag{3}$$

into the system (2) with $a = 3/2$, we obtain recursion relations for the coefficients v_n and w_n of (3), and then analyze the compatibility of those recursion relations at the resonances. We find that the recursion relations are compatible at the resonances 0, 1, 4 and 6, where the arbitrary functions $w_0(t)$, $w_1(t)$, $v_4(t)$ and $v_6(t)$ appear in the expansions (3), respectively; the resonance -1, as usual, corresponds to the arbitrariness of ψ in $\phi = x + \psi(t)$. However, at the highest resonance, $r = 8$, where the arbitrary function $w_8(t)$ enters the general solution, we encounter the following compatibility condition which restricts the arbitrary functions appeared at lower resonances:

$$300w_1{v_4}' - 7w_1\phi'\phi'' + 6{w_0}'\phi'' = 0, \tag{4}$$

where the prime denotes $\partial/\partial t$.

The appearance of the compatibility condition (4) means that the Laurent type expansions (3) do not represent the general solution of the studied system, and we have to modify the expansion for w by introducing logarithmic terms, starting from the term proportional to $\phi^6 \log \phi$. This non-dominant logarithmic branching of solutions is a clear symptom of non-integrability. Thus, the case $a = 3/2$ of the system (2) (and of the system (1), equivalently) fails the Painlevé test.

Conclusion

We have shown that, contrary to what was claimed by Hirota, Hu and Tang in [1], the system of coupled KdV equations (1) with $a = 3/2$ does not pass the Painlevé test for integrability. We have to note, moreover, that the singularity analysis of coupled KdV equations has been done in the papers [4] and [5], published prior to [1]. For instance, the integrable cases $a = 1$ and $a = 1/2$ of the system (1) can be found in [5] as the systems (vi) and (vii) which passed the Painlevé test well, whereas the case $r_1 = 1$ in Section 2.1.3 of [5] tells that the system (1) with $a = 3/2$ must fail the Painlevé test for integrability.

References

[1] R. Hirota, X.-B. Hu, X.-Y. Tang. A vector potential KdV equation and vector Ito equation: soliton solutions, bilinear Bäcklund transformations and Lax pairs. J. Math. Anal. Appl. 288:326–348 (2003).

[2] J. Weiss, M. Tabor, G. Carnevale. The Painlevé property for partial differential equations. J. Math. Phys. 24:522–526 (1983).

[3] M. Jimbo, M. D. Kruskal, T. Miwa. Painlevé test for the self-dual Yang–Mills equation. Phys. Lett. A 92:59–60 (1982).

[4] A. (Kalkanli) Karasu. Painlevé classification of coupled Korteweg–de Vries systems. J. Math. Phys. 38:3616–3622 (1997).

[5] S. Yu. Sakovich. Coupled KdV equations of Hirota–Satsuma type. J. Nonlinear Math. Phys. 6:255–262 (1999); arXiv solv-int/9901005.